ESTIMATION OF SPECTRAL CHARACTERISTICS
FOR A CLASS OF NON-STATIONARY TIME SERIES

by

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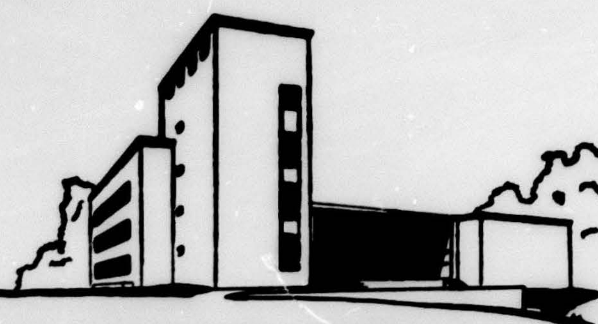

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1. Summary

This paper deals with time series which are stationary in the mean but which have a nonstationary convariance function which is described by a certain parametric model. The non-stationarity results from changes over time of the parameters in the model.

The first part of this paper discusses the concepts of stationarity, covariance, and the spectrum. The concept of temporal spectrum is introduced and applied to the analysis of the nonstationary time series.

The second part deals with the parametric model. Statistics related to the rate of zero-axis crossings and the rate of relative maxima and minima of the time series are used to estimate the parameters in the model. These estimates are easy to compute from discrete observations of the time series. Expressions for the variances of these estimators are given as a function of the parameters and the sampling period.

2. The Spectrum

In the analysis of time series data, a major problem is the characterization of the time series by means of some parameters or functions. For example, if $X(t)$ denotes the signal picked up by the antenna of an FM radio, the AGC (automatic gain control)circuit estimates the average power $E(X^2(t))$ and corrects the gain to equalize variations over time in this parameter. In order for this system to be effective, the average power parameter must be varying slowly enough so that we can obtain reasonably accurate estimates of the parameter given a finite record of $X(t)$. We must be able to distinguish changes in the characterization parameters from transient changes, just "ordinary noise,"of the time series.

The quest for slowly varying basic parameters leads us to the concept of stationarity. A time series $X(t)$ is called strictly stationary if for all n, the joint distribution of $\{X(t_1), \ldots, X(t_n)\}$ is the same as that for $\{X(t_1 + h), \ldots, X(t_n + h)\}$ for all h, Doob [2]. $X(t)$ is called weakly stationary if the density of $X(t)$ is the same for all t, and for all t_1 and t_2, the joint density of $(X(t_1), X(t_2))$ depends only on $|t_2 - t_1|$. Then the power spectrum $S(f)$ and the autocovariance function $R(\tau)$ are functions associated with $X(t)$, where

(1)
$$S(f) = \int_{-\infty}^{\infty} exp(2\pi f \tau) R(\tau)\, d\tau$$
$$R(\tau) = E\,(X(t+\tau)\, X(t))$$

With the assumption that $\int_0^{\infty} |\, R(\tau)\, |\, d\tau < \infty$, we then have

$$R(\tau) = 2 \int_0^{\infty} \cos 2\pi f \tau\, S(f) df$$

and by differentiating and setting $\tau = 0$, we have

(2)
$$R''(0) = -(2\pi)^2 \int f^2 S(f) df$$
$$R^{(4)}(0) = (2\pi)^4 \int f^4 S(f) df$$

If we normalize $X(t)$ such that $E(X(t)) = 0$ and $E(X^2(t)) = 1$, then the even function, $R(\tau) = R(-\tau)$, is called the autocorrelation function and $S(f)$ is called the power spectral density. The word density applies since $\int S(f) df = E(X^2(t))$ from (1) and (2, and thus $\int S(f) df = 1$.

Given a finite record of $X(t)$, of length T seconds, there are several ways to obtain an estimate of the spectrum, see Bartlett [1] and Grenander and

Rosenblatt [3]. These digital procedures or a good analog spectral analyser gives, as a function of the observations in the record of $X(t)$, a random function $S_T(f)$ such that for any band (f_0, f_1), (Middleton [14])

(3) $$\int_{f_0}^{f_1} S_T(f)df \xrightarrow{P} \int_{f_0}^{f_1} S(f)df \quad \text{as } T \to \infty .$$

It can be shown that if we pass $X(t)$ through a linear, time invariant filter which filters out (analogous to an optical filter) all frequencies except those in the band (f_0, f_1), then with $X_{f_0, f_1}(t)$ denoting this filtered time series,

(4) $$\int_{f_0}^{f_1} S(f)df = E\{X^2_{f_0, f_1}(t)\} .$$

The expected value of the square of a time series is called the average power of that series. Thus (4) shows that the area under $S(f)$ for f in (f_0, f_1) gives the average power of $X(t)$ in the band, and (3) shows that the integral of $S_T(f)$ over (f_0, f_1) is a consistent estimate (Wilks [13]) of this average power.

Moreover, if $X(t)$ is ergodic (Doob [2]) then with probability one

(5)
$$E\{X(t)\} = \lim_{T \to \infty} T^{-1} \int_0^T X(t)dt$$
$$R(\tau) = \lim_{T \to \infty} T^{-1} \int_0^{T-\tau} X(t+\tau)X(t)\,dt$$

By using the Chebychev inequality and the Gaussian nature of $X(t)$, we have from (5)

$$(6) \qquad R_T(\tau) = T^{-1}\int_0^{T-\tau} X(t+\tau)X(t)dt$$

$$= R(\tau) + 0_p \quad (T^{-1})$$

where R_T is called the sample autocorrelation of $X(t)$ and $0_p(T^{-1}) \rightarrow 0$ in probability. Thus from (4) and (6) we have

$$(7) \qquad T^{-1} \int_0^T X^2_{f_0,f_1} \quad (t)dt = \int_{f_0}^{f_1} S(f)df + 0_p \quad (T^{-1})$$

which suggests that we use the above time average of $X^2_{f_0,f_1}$, the filtered time series, as an estimator $S_T(f)$ of the average power for f in the band (f_0, f_1).

To do spectral estimation as above, using analog methods, requires a set of narrowband filters covering a large frequency range.

But suppose that $X(t)$ is stationary in the mean but not in the covariance, or more specifically that although $E(X(t)) = 0$ for all t, the joint density of $(X(t), X(t+\tau))$ depends on t. In this case, $S_T(f)$ is an estimate of the average power for f is a band, but averaged over distribution changes during the sampling period. The basic difficulty is that we must distinguish variations in $X(t)$ due to "ordinary noise" versus changes over time in the structure of the time series. In order to handle the problem we would like to characterise the nonstationary $X(t)$ by some generalisation of the power spectrum, such as proposed by Silverman [9]. Other extensions of spectral methods to nonstationary processes are given by Page [11] and Lampard [12]. However, we will treat the problem in somewhat of a heuristic manner.

We will restrict ourselves to stationary Gaussian time series $X(t)$ with $E(X(t)) = 0$ and $E(X^2(t)) = 1$ for all t, and such that given a record of length T units the joint density of $(X(t_1), X(t_2))$ is a function of $|t_2 - t_1|$ for t_1 and t_2 in the T unit period, i.e. $X(t)$ is weakly stationary during that period.

Furthermore, assume that there exists a $\tau^* < T$ for $|t_2 - t_1| > \tau^*$ there is a constant A and an integer m such that

$$(8) \qquad \left| E(X(t_2)X(t_1)) \right| < A \left| t_2 - t_1 \right|$$

This means that for lags greater than τ^* the time series is approximately uncorrelated.

Now define a function $S(f)$, called the temporal spectral density, such that as a generalisation of the definitions (1) and (2)

$$(9) \qquad S(f) = 2\int_0^T \cos 2\pi f \tau \, R(\tau) d\tau$$

where for $|\tau| < T$

$$(10) \qquad R(\tau) = E(X(t + \tau)X(t))$$

It can be shown using (7), (8), (9), and (10) that

$$(11) \qquad \int_{f_0}^{f_1} S(f)df = E(X^2_{f_0,f_1}(t)) + O(T^{-1}) = \int_0^T X^2_{f_0,f_1}(t)dt + O_p(T^{-1})$$

where X_{f_0,f_1} is the result of a linear narrowband filtering operation on $X(t)$ for the (f_0, f_1) band.

Moreover for $|\tau| < \tau^*$ by taking the inverse cosine transform,

$$(12) \qquad R(\tau) \triangleq 2\int_0^{\infty} \cos 2\pi f \tau S(f)df$$

for the temporal spectrum $S(f)$, and given the condition that $E(X^2(t)) = 1$, we then have

$$\int_{-\infty}^{\infty} S(f)df \triangleq 1 \quad .$$

Suppose we have a finite record of $X(t)$ during which $X(t)$ is stationary and we wish to estimate the temporal spectral density $S(f)$ for f in some frequency band. We could use the method of narrowband filtering (see (7) and (11)) to estimate $S(f)$. However, in many applications the period of stationary is so short as to make the spectral estimates which are functions of observations taken in the period, quite inaccurate. For example, the ambient sea noise, as measured by hydrophones placed in the deep ocean, is often quite nonstationary due to sudden changes in the paths of propagation of sounds in the ocean. If we wish accurate spectral estimates for f in a relatively wide band, in many applications we have to take a record of $X(t)$ which is longer than the period of stationarity, and thus the temporal spectrum approach is not meaningful since we have averaged over a change in structure.

However, suppose we know from theoretical considerations, or from past experimentation, that when the time series has a period of stationarity, the temporal spectral density $S(f)$ is one of a certain parametric family of density functions. For example, suppose we can say that

$$S(f) = \frac{\lambda}{2\Gamma(r)} (\lambda |f|)^{r-1} e^{-\lambda |f|} \tag{13}$$

where r and λ are two unknown parameters. In this case we have $S(f)$ described by a family of Gamma densities (Figures 1 and 2). For different stationarity periods, r and λ take different values. We thus have reduced the nonstationarity of $X(t)$ to the changes of two (or in general a finite set) parameters in the temporal spectrum. From (12) we see that the auto-correlation function $R(\tau)$ is a nonstationary function of these time varying parameters.

However, a <u>stationary</u> $X(t)$ with a spectral density as in (13) is deterministic (Doob [2]) since

$$\int_0^\infty (1 + f^2)^{-1} \log S(f) df = - \infty$$

This means that given a time t^*, the random variable $X(t^*)$ can be estimated with zero variance by a random variable $Y(t^*)$ which is the result of a certain linear operation on the record of the infinite past, i.e. on $X(t)$ for all $t < t^*$. But since we are dealing with nonstationary time series, results based upon observations over the infinite past do not seem to be relevant.

3. Parametric Model for the Temporal Spectrum

Suppose we make discrete observations of a Gaussian time series $X(t)$ during a stationary period where the temporal spectrum $S(f)$ is a Gamma density parameterized by r and λ - as is given by (13), i.e.

$$(13)^* \qquad S(f) = \frac{\lambda}{2\Gamma(r)} (\lambda f)^{r-1} e^{-\lambda f} \qquad f > 0$$

We wish to estimate r and λ from the discrete observations. The estimators are of no use if, in order to obtain relatively accurate estimates of r and λ, we need a sampling period longer than the period of stationarity of X. Any estimators r and λ based on observations taken over time from $X(t)$ will have two components of randomness, the first due to the stationary random fluctuations over time of $X(t)$, and the second due to the changes in the covariance of $X(t)$ which recur from time-to-time.

The rate of zero-axis crossings of $X(t)$ in an interval $(0,T)$ is a consistent estimator of $\beta = 2[\int f^2 S(f)df]^{1/2}$ as $T \to \infty$, and the rate of relative maxima and minima of X in $(0,T)$ is a consistent estimator of $\beta^* = 2[\int f^4 S(f)df / \int f^2 S(f)df]^{1/2}$. We will express r and λ as functions of β and β^*, $r = f(\beta,\beta^*)$ and $\lambda = g(\beta,\beta^*)$, and estimate r and λ by $r = f(\hat{\beta},\hat{\beta}^*)$ and $\hat{\lambda} = g(\hat{\beta},\hat{\beta}^*)$ where $\hat{\beta}$ is related to the rate of zero-crossings and $\hat{\beta}^*$ is related to the rate of maxima and minima, but they are easy to compute from discrete clipping of $X(t)$, which we will describe later. The consistency of the estimators is not important due to the nonstationarity of $X(t)$. For the estimators to be of real value in this context, their variances should decrease relatively rapidly with increase in the period of sampling. In this work we will obtain the variances for the estimators of β and β^* as functions of r, λ, and T. For the case of low frequencies, that is for the mean frequency $\int |f| S(f)df = r/\lambda$ in the range 20 cps to 40 cps, calculations (Hinich [4]) show that the standard deviation of the estimates of β and β^* is less than 10% of β for a sampling period of <u>two seconds</u> and less than 5% of β for a sampling period of <u>ten seconds</u>.

Suppose we sample $X(t)$ in a discrete manner by observing every τ time units whether the time series is positive or negative. If X is positive, we mark a 1, if negative we mark a -1, i.e. we define a discrete time series X^* where for each integer k

$$(14) \qquad X^*(k\tau) = \begin{cases} 1 & \text{if } X(k\tau) \geq 0 \\ -1 & \text{if } X(k\tau) < 0 \end{cases}$$

This is simply the discrete version of infinite clipping (Lawson and Uhlenbeck [5]). With the advent of high speed sampling techniques using digital electronics, it is a standard procedure to subject a random process to infinite clipping in order to reduce the data and put it into a convenient form (a string of binary numbers) for real-time analysis. The estimators for this parametric model have the useful property that they are easy to obtain from the clipped process.

Now let $N(T,\tau)$ denote the number of transitions of X^* during the interval $(0,T)$ from -1 to 1 and from 1 to -1. From Lemma 3 we have

$$(15) \qquad \lim_{T\to\infty} N(T,\tau)/T = \beta + 0\,(\tau^2)$$

where $\beta^2 = 4\int_{-\infty}^{\infty} f^2\, S(f)df$ and thus we have from (13),

$\beta^2 = 4\,\lambda^{-2}\, r(r+1)$. Thus from (15), we have

$$(16) \qquad \lim_{T\to\infty} N(T,\tau)/T = \frac{2}{\lambda}\,[r(r+1)]^{1/2} + 0\,(\tau^2)$$

The rate of binary transitions of X^* - $N(T,\tau)/T$ - is intimately conncected with the rate of zero-axis crossings of X provided the process X is 'well-behaved.' To argue heuristically, select τ sufficiently small such that in any τ unit interval, the probability of two or more axis-crossings can be neglected. Since either there is one crossing or no crossing in each interval $(k\tau, k\tau + \tau)$, the number of axis-crossings of X in the broad interval $(0,T)$ is just $N(T,\tau)$ and thus $N(T,\tau)/T$ is the rate of axis-crossings of the time series.

In order to give the variance of $N(T,\tau)/T$ we need the autocorrelation $R(\tau)$ of $X(t)$. By taking the cosine transformation of (13), we have

$$(17) \qquad R(\tau) = \mathrm{Re}\,\left(1 - \frac{2\pi i \tau}{\lambda}\right)^{-r}$$

where i is the complex unit and $\mathrm{Re}\ z$ is the real part of a complex z. We shall restrict r to be an integer for computational convenience in obtaining the sampling variances of several estimators.

Using (17) in Lemma 4 along with the relations

(18)
$$R'(\tau) = -2\pi\frac{r}{\lambda}\ \mathrm{Im}\left(1-\frac{2\pi i\tau}{\lambda}\right)^{-(r+1)}$$

$$R''(\tau) = -(2\pi)^2\frac{r(r+1)}{\lambda^2}\ \mathrm{Re}\left(1-\frac{2\pi i\tau}{\lambda}\right)^{-(r+2)}$$

where R' is the first derivative of R with respect to τ and R'' is the second derivative, and changing variables $t = 2\pi\frac{r}{\lambda}x$, we have

(19)
$$\lim_{\tau\to 0} \mathrm{Var}\,[N(T,\tau)/T]$$

$$= \frac{\beta^2}{2T^*}\left[\frac{2}{\pi}\int_0^{2\pi T^*}\left(1-\frac{t}{2\pi T^*}\right)a_r(t)dt + \left(\frac{r}{r+1}\right)^{1/2}\right]$$

where

$$T^* = \frac{r}{\lambda}\ T \qquad\qquad \beta = \frac{2}{\lambda}\ [r(r+1)]^{1/2}$$

(20a)
$$a_r(t) = \frac{[b_r(t)]^{1/2}}{1-e_r^2(t)}\ [1 + c_r(t)\arctan c_r(t)] - 1$$

(20b)
$$b_r(t) = [1-e_r^2(t)]\,[1-e_{r+2}^2(t)]$$

$$- 2\frac{r}{r+1}d_r^2(t)\,[1-e_r(t)\,e_{r+2}(t)]$$

$$+ \left(\frac{r}{r+1}\right)^2 d_r^4(t)$$

$$(20c)\qquad c_r(t) = \frac{\frac{r}{r+1}\, e_r(t) d_r^2(t) - e_{r+2}(t)\,[1-e_r^2(t)]}{[1-e_r^2(t)]^{1/2}\,[b_r(t)]^{1/2}}$$

$$(20d)\qquad d_r(t) = r^{-1}R'(rx) = \frac{\sum_{j=0}^{r^*} \binom{r+1}{2j+1} (-1)^j \left(\frac{t}{r}\right)^{2j+1}}{[1+(\frac{t}{r})^2]^{r+1}}$$

$$(20e)\qquad e_r(t) = R(rx) = \frac{\sum_{j=0}^{r^*} \binom{r}{2j} (-1)^j \left(\frac{t}{r}\right)^{2j}}{[1+(\frac{t}{r})^2]^{r}}$$

where

$$r^* = \begin{cases} \frac{r}{2} & \text{if } r \text{ is even} \\ \frac{r-1}{2} & \text{if } r \text{ is odd} \end{cases}$$

Remember that $r/\lambda = 2\int_0^\infty f\ S(f)df$ which is the "mean" frequency of the temporal spectrum ,

We will now discuss the estimator related to the number of maxima and minima per unit interval. Define the derivative process

$$(21) \qquad X'(t) = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}$$

where the convergence is in the mean (Doob [2]). If X is a Gaussian process with an autocorrelation R such that

$$(22) \quad R(\tau) = 1 + R''(0)\frac{\tau^2}{2} + R^{(4)}(0)\frac{\tau^4}{4!} + 0(\tau^5)$$ then X' exists almost everywhere and is also Gaussian (Doob [2]). Moreover, it can be easily shown that the autocovariance of X' is $-R''(\tau)$, where R'' is the second derivative of R with respect to τ . If we normalise X' by defining a process

$$(23) \qquad Y(t) = [-R''(0)]^{-1/2} X'(t)$$

we have $E\{Y(t)\} = 0$, $E\{Y^2(t)\} = 1$ and

$$(24) \quad E\{Y(t+\tau)Y(t)\} = \frac{R''(\tau)}{R''(0)}$$

If we let D(f) denote the spectral density of Y, we have from (2) and (24)

$$(25) \qquad \int f^2 D(f)df = (2\pi)^{-2} [R''(0)]^{-1} R^{(4)}(0)$$
$$= \int f^4 S(f)df / \int f^2 S(f)df$$

The process X has a relative maxima or minima at t if and only if $X'(t) = 0$, and thus if and only if $Y(t) = 0$. If we let $M(T, \tau)$ be the number of transitions from 1 to -1 and from -1 to 1 of the discrete process

$$(26) \qquad Y^*(k\tau) = \begin{cases} 1 \text{ if } Y(k\tau) \geq 0 \\ -1 \text{ if } Y(k\tau) < 0 \end{cases}$$

for $k = 0,1,\ldots,T/\tau$, then for small τ, M is approximately the number of relative maxima or minima in the interval (0,T). Therefore $M(T, \tau)/T$ is essentially the rate of maxima and minima of X. Using (23), (25), and (26) in Lemma 3, we have

$$(27) \qquad \lim_{T\to\infty} M(T,\tau)/T = \beta^* + 0(\tau^2)$$
$$= 2[\int f^4 S(f)df / \int f^2 S(f)df]^{1/2} + 0(\tau^2)$$

where β^* is the expected number of relative maxima or minima per unit interval (Rice [7]).

However we do not have to differentiate X in order to estimate β^*. Let us define another discrete process

$$(28) \qquad X^{**}(k\tau) = \begin{cases} 1 \text{ if } X((k+1)\tau) \geq X(k\tau) \\ -1 \text{ if } X((k+1)\tau) < X(k\tau) \end{cases}$$

From the definition of X' and (28), it is fairly clear that the rate of binary transitions of X^{**} for sufficiently small τ is also $M(T,\tau)/T$.

Applying (27) to the case of a Gamma spectrum, we have

$$(29)\qquad \lim_{T\to\infty} M(T,\tau)/T = \beta^* = \frac{2}{\lambda}\,[(r+2)\ (r+3)\ ]^{1/2}$$

since from (13)

$$\int f^4 S(f)df = \lambda^{-4} r(r+1)(r+2)(r+3)$$

We can obtain the variance of $M(T,\tau)/T$ fairly easily based on the derivation of (19). Applying (18) to (24), we have for the autocorrelation of Y,

$$(30)\qquad E\{Y(t+\tau)Y(t)\} = \mathrm{Re}(1-\frac{2\pi i\tau}{\lambda})^{-(r+2)}$$

which, from (17), is just the autocorrelation of X with r replaced by r+2. Setting — $T^{**} = \frac{r+2}{\lambda}T$ and $\beta^* = 2\lambda^{-1}[(r+2)(r+3)]^{1/2}$ (just β with r+2 instead of r), we have

$$(31)\qquad \lim_{\tau\to 0} \mathrm{Var}[M(T,\tau)/T]$$

$$= \frac{\beta^{*2}}{2T^{**}}\,[\frac{2}{\pi}\int_0^{2\pi T^{**}} (1-\frac{t}{2\pi T^{**}})\ a_{r+2}(t)dt$$

$$+ (\frac{r+2}{r+3})^{1/2}\ ]$$

which is the variance of $N(T,\tau)/T$ as given by (19) and (20), but where r is replaced by r+2.

We will now propose estimators or r and λ. From (16) and (29) we have

$$\frac{\beta^*}{\beta} = \left[\frac{(r+2)(r+3)}{r(r+1)}\right]^{1/2} > 1$$

which yields

$$r = \frac{5\beta^2-\beta^{*2} + [\beta^{*4} + 14\beta^{*2}\beta^2 + \beta^4]^{1/2}}{2(\beta^{*2} - \beta^2)} \tag{32}$$

Let the estimator of r by $\hat{r}$ where

$$\hat{r} = \frac{5N^2-M^2 + [M^4 + 14\, M^2N^2 + N^4]^{1/2}}{2(M^2-N^2)}$$

for $N = N(T,\tau)$ and $M = M(T,\tau)$. Thus from (16) and (29) we see that $\hat{r}$ is a consistent estimator of r. Since r is restricted to be an integer, choose [r] as the estimate of r where [r] is the closest integer to $\hat{r}$. [r] is also a consistent estimator of r given the stationarity of X. But since X is in reality nonstationary, the consistency of the estimator is indeed a weak property. The usefulness of r will be mainly determined by the variance of $\hat{r}$, which we do not give since we have not developed the covariance of N and M. However, if in a certain application for a specified pair of parameters r and λ, the variances of N/T and M/T are reasonably small, then it is worthwhile to find the exact variance of $\hat{r}$ as a function of T by Monte Carlo techniques. In the case of mean frequencies 20 cps < r/λ 40 cps, [r] had a 5% standard deviation for T of less than a minute.

Again from (16),

$$(33) \qquad \hat{\lambda} = \frac{2T}{N(T,\tau)} \left[\hat{r}(\hat{r}+1)\right]^{1/2}$$

is a consistent estimator of λ.

Both $\hat{r}$ and $\hat{\lambda}$ are relatively easy to compute from a sequence of discrete observations on the clipped version of the time series $X(t)$. Since in a great many applications dealing with the above type of nonstationary time series, the process is first subjected to infinite clipping, the estimation method given above offers a practical procedure for dealing with the changes of the harmonic power levels over time.

4. Ergodic Results for the Rate of Axis-Crossings

Let $N(T)$ be the number of zero-axis crossings of $X(t)$ in the interval $(0,T)$ and assuming that $E\{X(t)\} = 0$ and $E\{N(T)\} < \infty$, define

$$(34) \qquad \beta = \frac{E\{N(T)\}}{T}$$

At first glance it looks as if β is a function of T. If $X(t)$ is strictly stationary then for all T, the expected number of zero-crossings in any T unit interval is the same as $E\{N(T)\}$. Now if we choose any unit of time τ and let K be the positive integer $K = T/\tau$, then from (34)

$$\beta = (K\tau)^{-1} \sum_{k=1}^{K} E\{N(\tau)\} = \tau^{-1} E\{N(\tau)\}$$

and thus β is not a function of T.

Rice [7] shows that if X is Gaussian, $\beta = 2[\int f^2 S(f)df]^{1/2}$ where S(f) is the spectrum of X. However, in Rice's derivation and also in McFadden's [6], it is assumed that a τ can be found sufficiently small such that in the interval $(t, t + \tau)$, the probability of exactly one axis-crossing is simply β and the probability of two or more crossings can be neglected. This assumption should be carefully considered since if X is white noise, there are infinitely many zeros in any interval. We will now give three simple lemmas which will show that N(T)/T is a consistent estimate of β; that the second moment of the spectrum of a Gaussian process (if it exists) can be estimated from discrete clipping; and then give the variance of the consistent estimator.

<u>Lemma 1</u>: Let X(t) be a strictly stationary and metrically transitive (Doob[2]) random process and N(T) be the number of axis-crossings in the interval (0,T). Assume that $E\{N(T)\}$ exists. Then as $T \to \infty$

$$N(T)/T \to \beta$$

with probability one.

Proof: Express T in integer multiples of a convenient unit τ, i.e. $T = K\tau$ for a positive integer K. Let N_k denote the number of axis-crossings of X in the interval $((k-1)\tau, k\tau)$ for $k=1,\ldots,K$ Thus we have

$$(35) \qquad N(T) = \sum_{k=1}^{K} N_k$$

Since X is strictly stationary and metrically transitive, so is the discrete process $\{N_k\}$. Thus we have the ergodic result that as $K \to \infty$, with probability one

(36) $$K^{-1} \sum_{k=1}^{K} N_k \rightarrow E\{N_k\}$$

But from (34), $E\{N_k\} = \beta\tau$ and thus by using (35) in (36) and dividing by τ we have the desired result.

Lemma 2: Let $X(t)$ be a strictly stationary and metrically transitive random process. Given a τ, define the discrete process

$$X^*(k\tau) = \begin{cases} 1 & \text{if } X(k\tau) \geq 0 \\ -1 & \text{if } X(k\tau) < 0 \end{cases}$$

for the integers $k = 0, 1, \ldots$. Let $N(T, \tau)$ denote the number of transitions of X^* from 1 to -1 and from -1 to 1 in the interval $(0,T)$ where $T = K\tau$. Let $R^*(\tau) = E\{X^*(k\tau)\, X^*((k+1)\tau)\}$ which does not depend on k since X^* is strictly stationary and metrically transitive. Then

$$N(T, \tau)/T \rightarrow (2\tau)^{-1} [1 - R^*(\tau)]$$

with probability one as $T \rightarrow \infty$.

Proof: By the ergodicity of X^*, the sample autocorrelation

(37) $$R^*_T(\tau) = \frac{1}{K} \sum_{k=0}^{K-1} X^*((k+1)\tau)\, X^*(k\tau)$$

converges with probability one to the autocorrelation R^* as $K = T/\tau \rightarrow \infty$. Applying the definition of X^* to (37) we have

$$R_T^*(\tau) = K^{-1}[K-2N(T,\tau)]$$

and thus by dividing by τ in the convergence $R_T^* \to R^*$, we have the desired result.

Now suppose X(t) is stationary Gaussian process where $E\{X(t)\} = 0$ and $E\{X^2(t)\} = 1$. It then possesses the properties required for the lemmas. Thus $N(T)/T \to \beta$ with probability one as $T \to \infty$ and similarly for Lemma 2. From Slepian [8] we can show Rice's [7] result that

(39) $$\beta = \pi^{-1}[-R''(0)]^{1/2} = 2\left[\int f^2 S(f)df\right]^{1/2}$$

given that R is twice continuously differentiable at $\tau = 0$.

But Lawson and Uhlenbeck [5] show that

(40) $$R^*(\tau) = 2\pi^{-1}\arcsin R(\tau)$$

By series expansion of arcsin in powers of τ, we can show from (40)

(41) $$\frac{1-R^*(\tau)}{2\tau} = \pi^{-1}[-R''(0)]^{1/2} + \frac{[R''(0)]^2 - R^{(4)}(0)}{[-R''(0)]^{1/2}}\,\frac{\tau^2}{24\pi} + 0(\tau^3)$$

where now we assume that R has the expansion

(42) $$R(\tau) = 1 + R''(0)\frac{\tau^2}{2} + R^{(4)}(0)\frac{\tau^4}{4!} + 0(\tau^5)$$

From (39), (41) and Lemma 2 we have

<u>Lemma 3</u>: Suppose $X(t)$ is a stationary Gaussian process with an auto-correlation function $R(\tau)$ as given by (42). Then with probability one

$$\lim_{\tau \to 0} \lim_{T \to 0} N(T, \tau)/T = \beta$$

and

$$\lim_{T \to \infty} [N(T, \tau)/T - \beta] = \frac{2\pi^2}{3\beta} \tau^2 \{[\int f^2 S(f) df]^2 - \int f^4 S(f) df\} + 0(\tau^3)$$

where $S(f)$ is the spectral density of X.

We will not derive the variance of $N(T, \tau)/T$ as a function of T, τ, and β. From (38), Lemma 3, and $K = T/\tau$ we have

$$(43) \quad E[R_T^*(\tau)]^2 = 1 - 4\beta\tau + 4\, E[N(T, \tau)/T]^2 \, \tau^2 + 0(\tau^3)$$

and from (37)

$$(44) \quad E[R_T^*(\tau)]^2 = K^{-2} \sum_{k=1}^{K-1} \sum_{j=1}^{K-1} E\{X^*((k+1)\tau)\, X^*(k\tau) X^*((j+1)\tau) X^*(j\tau)\}$$

Let $D(x, \tau)$ denote the conditional probability that $X(t)$ crosses the axis in the interval $(x, x+\tau)$ given that it crossed in $(0, \tau)$. For Gaussian $X(t)$, Rice [7] derives a function $U(x)$ such that

$$D(x, \tau) = U(x)\tau + 0(\tau^2) \text{ and } \int_{-\varepsilon}^{\varepsilon} U(x) dx = 1 \text{ as } \varepsilon \to 0.$$

Since the probability of the process crossing the axis in the interval $(0, \tau)$ is $\beta\tau + O(\tau^2)$, we then have for $|k - j| = n \geq 1$

$$(45a)\qquad E\{X^*((k+1)\tau)X^*(k\tau)X^*((j+1)\tau)X^*(j\tau)\}$$

$$= U(n\tau)\,\tau(\beta\tau) - 2[1-U(n\tau)\tau(\beta\tau)]$$

$$+ \{1-U(n\tau)\quad(\beta\tau) - 2[1-U(n\tau)\tau(\beta\tau)]$$

$$+ O(\tau^3)$$

$$= 1 - 4\beta\tau + 4\,U(n\tau)\beta\,\tau^2 + O(\tau^3)$$

and for $k = j$

$$(45b)\quad E\{X^*((k+1)\tau)X^*(k\tau)X^*((j+1)\tau)X^*(j\tau)\} = 1$$

Setting (45) in (44) and summing, we have

$$(46)\quad E(R_T^*(\tau))^2 = 1 - 4\beta\tau + 8\beta\tau^2 T^{-1}\sum_{n=1}^{T/\tau}\left(1 - \frac{n\tau}{T}\right)U(n\tau)\tau$$

$$+ O(\tau^3)$$

and thus from (43)

$$(47)\quad E[N(T,\tau)/T]^2 = \frac{2\beta}{T}\sum_{n=1}^{T/\tau}\left(1 - \frac{n\tau}{T}\right)U(n\tau)\tau$$

Letting τ go to zero in (47) and since $\lim_{\varepsilon\to 0}\int_0^{\varepsilon} U(x)dx = \frac{1}{2}$

we have

$$(48)\qquad \lim_{\tau\to 0} \mathrm{Var}[N(T,\tau)/T] = \frac{2\beta}{T}\int_{0+}^{T}\left(1 - \frac{x}{T}\right)U(x)dx$$

$$+ \frac{\beta}{T} - \beta^2$$

If X is Gaussian with autocorrelation $R(\tau)$, using Rice's derivation of U(x), we have

Lemma 4:

$$\lim_{\tau \to 0} \mathrm{Var}[N(T, \tau)/T] = \beta^2 \left\{ \frac{2}{T} \int_{0+}^{T} (1-\tfrac{x}{T}) A(x)[1 + D(x) \arctan D(x)] dx + \frac{\pi}{T[-R''(0)]^{1/2}} - 1 \right\}$$

where

$$A(x) = [B^2(x) - C^2(x)]^{1/2} \, [1 - R^2(x)]^{-3/2}$$

$$B(x) = 1 - R^2(x) + [R''(0)]^{-1} [R'(x)]^2$$

$$C(x) = R(x) [R''(0)]^{-1} [R'(x)]^2 + R''(x) [R''(0)]^{-1} [1 - R^2(x)]$$

$$D(x) = C(x) [B^2(x) - C^2(x)]^{-1/2}$$

This is similar to the result of Steinberg, et.al. [10]

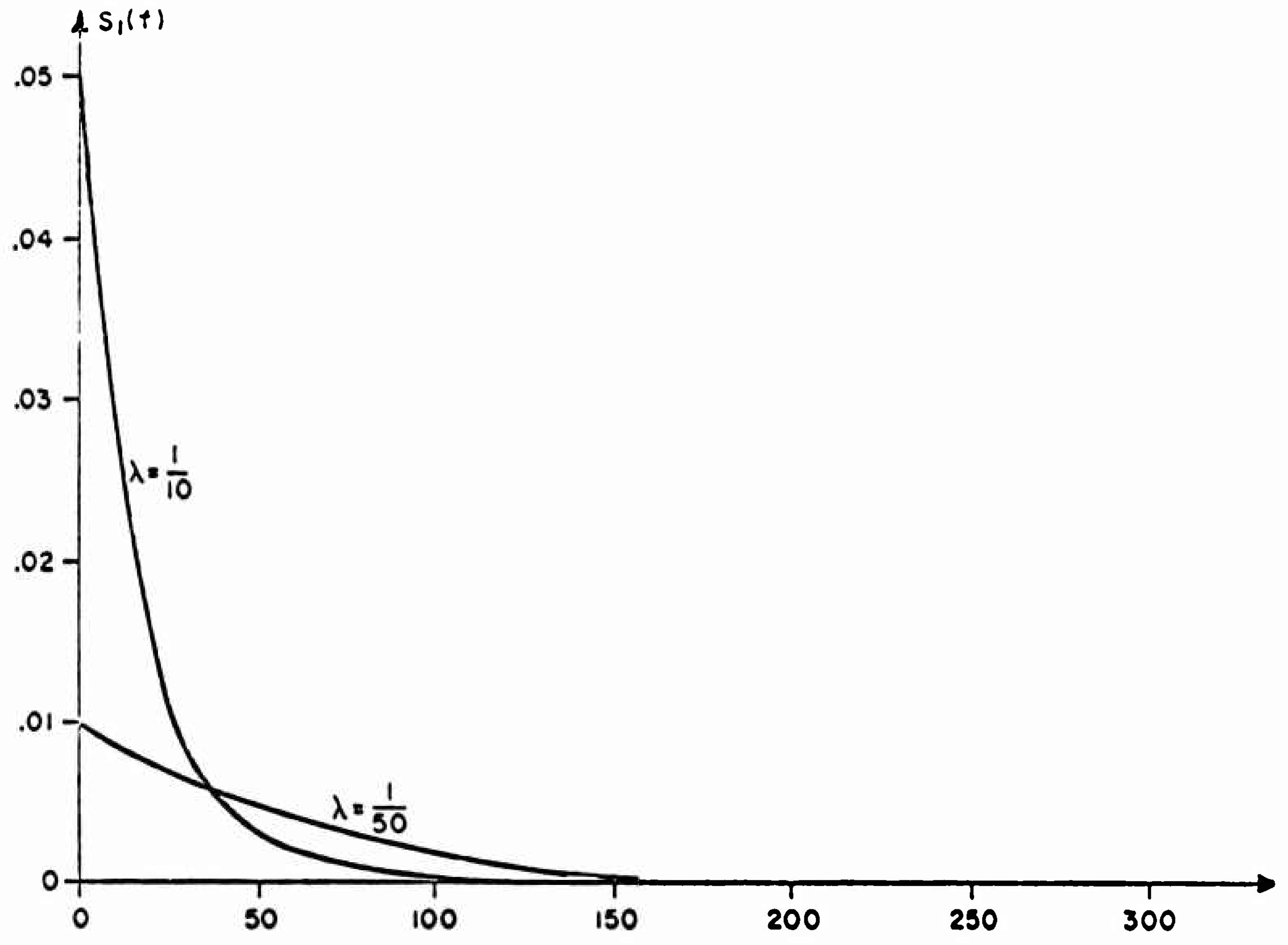

FIG. 1 GAMMA SPECTRUM FOR r = 1

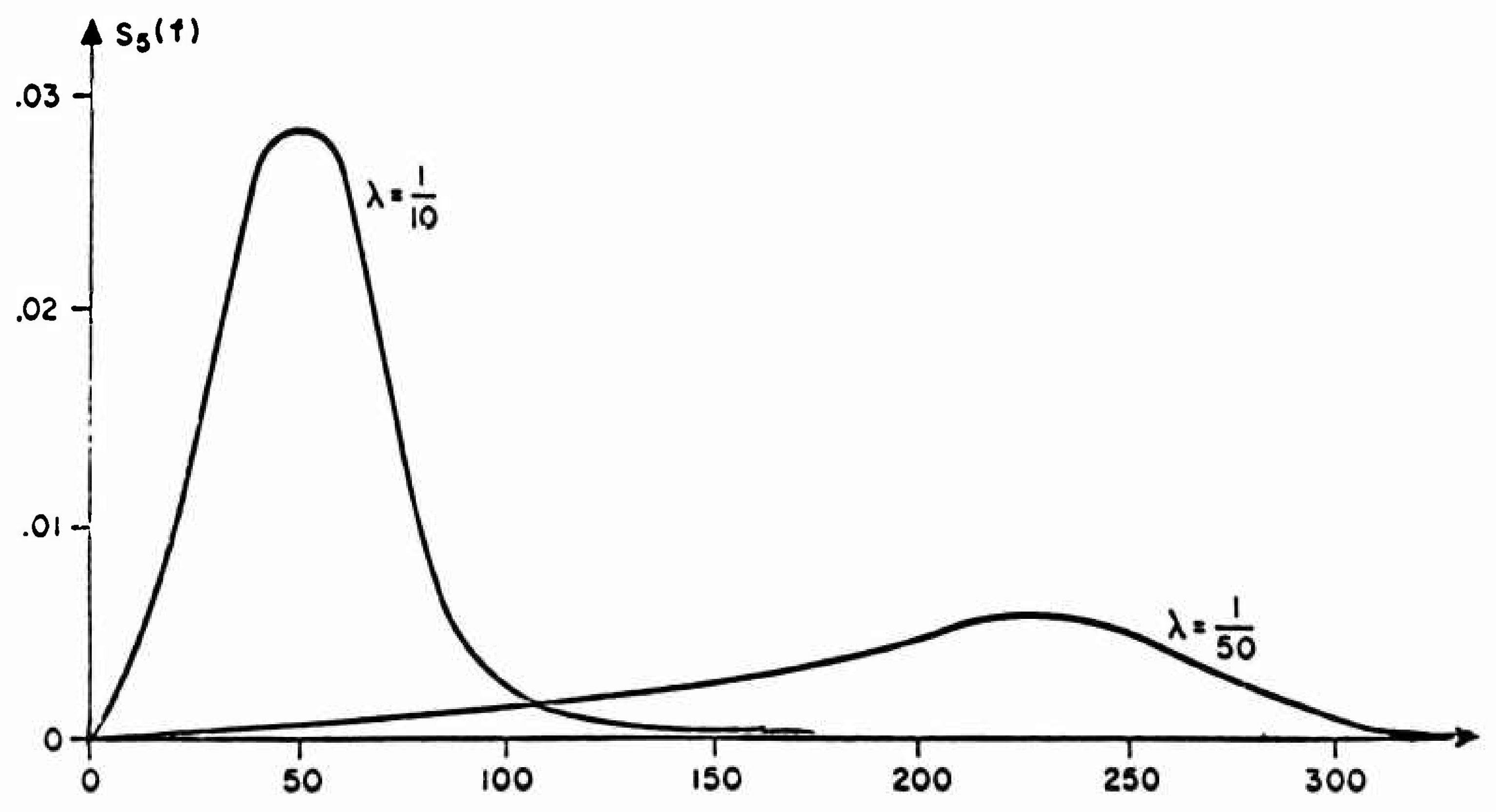

FIG. 2 GAMMA SPECTRUM FOR r = 5

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13. ABSTRACT

This paper deals with time series which are stationary in the mean but which have a nonstationary covariance function which is described by a certain parametric model. The non-stationarity results from changes over time of the parameters in the model.

The first part of this paper discusses the concepts of stationarity, covariance, and the spectrum. The concept of temporal spectrum is introduced and applied to the analysis of the nonstationary time series.

The second part deals with the parametric model. Statistics related to the rate of zero-axis crossings and the rate of relative maxima and minima of the time series are used to estimate the parameters in the model. These estimates are easy to compute from discrete observations of the time series. Expressions for the variances of these estimators are given as a function of the parameters and the sampling period.

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